Fixed Viewpoint Mirror Surface Reconstruction under an Uncalibrated Camera -Supplementary Material-

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1. Plücker Coordinates

A 3D line can be described by a Plücker matrix $\mathbf{L} = \mathbf{Q}\mathbf{P}^{\mathbf{T}} - \mathbf{P}\mathbf{Q}^{\mathbf{T}} =$

$$\begin{bmatrix} 0 & q_1p_2 - q_2p_1 & q_1p_3 - q_3p_1 & q_1p_4 - q_4p_1 \\ q_2p_1 - q_1p_2 & 0 & q_2p_3 - q_3p_2 & q_2p_4 - q_4p_2 \\ q_3p_1 - q_1p_3 & q_3p_2 - q_2p_3 & 0 & q_3p_4 - q_4p_3 \\ q_4p_1 - q_1p_4 & q_4p_2 - q_2p_4 & q_4p_3 - q_3p_4 & 0 \end{bmatrix}, \tag{1}$$

where $\mathbf{P} = [p_1\ p_2\ p_3\ p_4]^{\mathrm{T}}$ and $\mathbf{Q} = [q_1\ q_2\ q_3\ q_4]^{\mathrm{T}}$ are the homogeneous representations of two distinct 3D points on the line. Since $\mathbf{L}$ is skew-symmetric, it can be represented simply by a Plücker vector $\mathcal{L}$ consisting of its 6 distinct non-zero elements

$$\mathcal{L} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \end{bmatrix} = \begin{bmatrix} q_1p_2 - q_2p_1 \\ q_1p_3 - q_3p_1 \\ q_1p_4 - q_4p_1 \\ q_2p_3 - q_3p_2 \\ q_3p_4 - q_4p_3 \\ q_4p_2 - q_2p_4 \end{bmatrix}. \tag{2}$$

Dually, a matrix $\bar{\mathbf{L}}$ can be constructed from two distinct planes with homogeneous representations $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ as $\bar{\mathbf{L}} = \hat{\mathbf{Q}}\hat{\mathbf{P}}^{\mathrm{T}} - \hat{\mathbf{P}}\hat{\mathbf{Q}}^{\mathrm{T}}$. The *dual Plücker vector* can be constructed directly from $\bar{\mathbf{L}}$ or by rearranging the elements of $\mathcal{L}$ as

$$\bar{\mathcal{L}} = [l_5\ l_6\ l_4\ l_3\ l_1\ l_2]^{\mathrm{T}}. \tag{3}$$

Let $\mathbf{A} = [a_1\ a_2\ a_3]^{\mathrm{T}}$ and $\mathbf{B} = [b_1\ b_2\ b_3]^{\mathrm{T}}$ be two distinct 3D points in Cartesian coordinates. Geometrically, the line defined by these points can be represented by a direction vector $\boldsymbol{\omega} = (\mathbf{A} - \mathbf{B}) = [l_3\ \ -l_6\ l_5]^{\mathrm{T}}$ and a moment vector $\boldsymbol{\nu} = (\mathbf{A} \times \mathbf{B}) = [l_4\ \ -l_2\ l_1]^{\mathrm{T}}$, which define the line up to a scalar factor.

Two 3D lines $\mathcal{L}$ and $\mathcal{L}'$ can either be skew or coplanar. The geometric requirement for the latter case is that the dot product between the direction vector of the first line and the moment vector of the second line should equal the negative of the dot product between the direction vector of the second line and the moment vector of the first line. Let the two lines have direction vectors $\boldsymbol{\omega}$, $\boldsymbol{\omega}'$ and moment vectors $\boldsymbol{\nu}$, $\boldsymbol{\nu}'$, respectively. They are coplanar (i.e., either coincident or intersect) if and only if

$$\boldsymbol{\omega} \cdot \boldsymbol{\nu}' + \boldsymbol{\nu} \cdot \boldsymbol{\omega}' = 0 \Leftrightarrow \mathcal{L} \cdot \bar{\mathcal{L}}' = 0. \tag{4}$$

Note that a Plücker vector is not any arbitrary 6-vector. A valid Plücker vector must always intersect itself, i.e.,

$$\mathcal{L} \cdot \bar{\mathcal{L}} = 0 \Leftrightarrow det(\mathbf{L}) = 0. \tag{5}$$

2. Camera projection matrix vs. line projection matrix

Here we show the details for the conversion between camera projection matrix and the line projection matrix. First, consider the case of transforming a point projection matrix to its equivalent line projection matrix. Referring to Fig. 3 in our main paper, the plane $\mathbf{P}_{1*}\mathbf{X} = \mathbf{0}$ is defined by the camera center and the line $u = 0$ in the image plane. Similarly, $\mathbf{P}_{2*}\mathbf{X} = \mathbf{0}$

is defined by the camera center and the line $v = 0$ in the image plane. Finally, the plane equation $\mathbf{P}_{3*}\mathbf{X} = \mathbf{0}$ holds for all points with pixel coordinates $s = 0$. We can obtain the i-th row of $\mathcal{P}$ by the intersection of rows j and k of $\mathbf{P}$, i.e.,

$$\mathcal{P}_{i*}^{\mathrm{T}} = \begin{bmatrix}\rho_{i1}\\ \rho_{i2}\\ \rho_{i3}\\ \rho_{i4}\\ \rho_{i5}\\ \rho_{i6}\end{bmatrix} = (-1)^{(i+1)} \begin{bmatrix} p_{j3}p_{k4} - p_{j4}p_{k3}\\ p_{j4}p_{k2} - p_{j2}p_{k4}\\ p_{j2}p_{k3} - p_{j3}p_{k2}\\ p_{j1}p_{k4} - p_{j4}p_{k1}\\ p_{j1}p_{k2} - p_{j2}p_{k1}\\ p_{j1}p_{k3} - p_{j3}p_{k1}\end{bmatrix}, \tag{6}$$

where $\mathbf{P}_{i*}$ is the i-th row of $\mathbf{P}$, $\mathcal{P}_{i*}$ is the i-th row of $\mathcal{P}$, $i \neq j \neq k \in \{1,2,3\}$ and $j < k$. Note that $\mathcal{P}_{i*}^{\mathrm{T}}$ is the dual Plücker vector of $(-1)^{i+1}(\mathbf{P}_{j*}^{\mathrm{T}} \wedge \mathbf{P}_{k*}^{\mathrm{T}})$, i.e., the intersection of the j-th with the k-th row of $\mathbf{P}$. The sign here controls the order of intersection, i.e., $(\mathbf{P}_{j*}^{\mathrm{T}} \wedge \mathbf{P}_{k*}^{\mathrm{T}}) = -(\mathbf{P}_{k*}^{\mathrm{T}} \wedge \mathbf{P}_{j*}^{\mathrm{T}})$. Dually, we can obtain the i-th row of $\mathbf{P}$ by the intersection of rows j and k of $\mathcal{P}$, which results in the homogeneous plane

$$\begin{aligned}\mathbf{P}_{i*}^{\mathrm{T}} &= (-1)^{(i+1)} \begin{bmatrix}\boldsymbol{\omega}_j \times \boldsymbol{\omega}_k\\ \boldsymbol{\nu}_j \cdot \boldsymbol{\omega}_k\end{bmatrix}\\ &= (-1)^{(i+1)} \begin{bmatrix}\rho_{j5}\rho_{k6} - \rho_{j6}\rho_{k5}\\ \rho_{j5}\rho_{k3} - \rho_{j3}\rho_{k5}\\ \rho_{j6}\rho_{k3} - \rho_{j3}\rho_{k6}\\ \rho_{j4}\rho_{k3} + \rho_{j2}\rho_{k6} + \rho_{j1}\rho_{k5}\end{bmatrix},\end{aligned} \tag{7}$$

where again $i \neq j \neq k \in \{1,2,3\}$ with $j < k$, $\boldsymbol{\omega}_j$ is the direction vector of $\mathcal{P}_{j*}^{\mathrm{T}}$ and $\boldsymbol{\nu}_j$ is the moment vector of $\mathcal{P}_{j*}^{\mathrm{T}}$.

3. Plane Pose Estimation from Reflection Correspondences

As in Section III,we can obtain the solution space spanned by two vector basis, $\mathbf{d}_1$ and $\mathbf{d}_2$. $\mathbf{W}$ is then parameterized as

$$\mathbf{W} = \alpha\left(\mathbf{d}_1 + \beta\mathbf{d}_2\right), \tag{8}$$

where

$$\mathbf{d}_1 = \left(d_1^1,\ d_1^2,\ d_1^3,\ \cdots,\ d_1^{24}\right), \tag{9}$$
$$\mathbf{d}_2 = \left(d_2^1,\ d_2^2,\ d_2^3,\ \cdots,\ d_2^{24}\right). \tag{10}$$

The definition of $\mathbf{W}$, $\mathcal{A}$ and $\mathcal{B}$ are recollected in detail as follows

$$\mathbf{W} = \left[\mathcal{A}_{1*}\ \mathcal{A}_{2*}\ \mathcal{A}_{3*}\ \mathcal{B}_{1*}\ \mathcal{B}_{2*}\ \mathcal{B}_{3*}\ \mathcal{N}_{3*}\ \mathcal{M}_{3*}\right]^{\mathrm{T}}, \tag{11}$$
$$\mathcal{A} = \mathcal{N}_{3*}^{\mathrm{T}}\mathcal{M}_{1*} - \mathcal{N}_{1*}^{\mathrm{T}}\mathcal{M}_{3*}, \tag{12}$$
$$\mathcal{B} = \mathcal{N}_{3*}^{\mathrm{T}}\mathcal{M}_{2*} - \mathcal{N}_{2*}^{\mathrm{T}}\mathcal{M}_{3*}. \tag{13}$$

According to the element-wise equality between (8) and (11), we have

$$A_{11} = n_{31}m_{11} - n_{11}m_{31} = \alpha\left(d_1^1 + \beta d_2^1\right), \tag{14}$$
$$A_{12} = n_{31}m_{12} - n_{11}m_{32} = \alpha\left(d_1^2 + \beta d_2^2\right), \tag{15}$$
$$A_{13} = n_{31}m_{13} - n_{11}m_{33} = \alpha\left(d_1^3 + \beta d_2^3\right), \tag{16}$$
$$A_{21} = n_{32}m_{11} - n_{12}m_{31} = \alpha\left(d_1^4 + \beta d_2^4\right), \tag{17}$$
$$A_{22} = n_{32}m_{12} - n_{12}m_{32} = \alpha\left(d_1^5 + \beta d_2^5\right), \tag{18}$$
$$A_{23} = n_{32}m_{13} - n_{12}m_{33} = \alpha\left(d_1^6 + \beta d_2^6\right), \tag{19}$$
$$A_{31} = n_{33}m_{11} - n_{13}m_{31} = \alpha\left(d_1^7 + \beta d_2^7\right), \tag{20}$$
$$A_{32} = n_{33}m_{12} - n_{13}m_{32} = \alpha\left(d_1^8 + \beta d_2^8\right), \tag{21}$$
$$A_{33} = n_{33}m_{13} - n_{13}m_{33} = \alpha\left(d_1^9 + \beta d_2^9\right), \tag{22}$$
$$B_{11} = n_{31}m_{21} - n_{21}m_{31} = \alpha\left(d_1^{10} + \beta d_2^{10}\right), \tag{23}$$
$$B_{12} = n_{31}m_{22} - n_{21}m_{32} = \alpha\left(d_1^{11} + \beta d_2^{11}\right), \tag{24}$$
$$B_{13} = n_{31}m_{23} - n_{21}m_{33} = \alpha\left(d_1^{12} + \beta d_2^{12}\right), \tag{25}$$
$$B_{21} = n_{32}m_{21} - n_{22}m_{31} = \alpha\left(d_1^{13} + \beta d_2^{13}\right), \tag{26}$$
$$B_{22} = n_{32}m_{22} - n_{22}m_{32} = \alpha\left(d_1^{14} + \beta d_2^{14}\right), \tag{27}$$
$$B_{23} = n_{32}m_{23} - n_{22}m_{33} = \alpha\left(d_1^{15} + \beta d_2^{15}\right), \tag{28}$$
$$B_{31} = n_{33}m_{21} - n_{23}m_{31} = \alpha\left(d_1^{16} + \beta d_2^{16}\right), \tag{29}$$
$$B_{32} = n_{33}m_{22} - n_{23}m_{32} = \alpha\left(d_1^{17} + \beta d_2^{17}\right), \tag{30}$$
$$B_{33} = n_{33}m_{23} - n_{23}m_{33} = \alpha\left(d_1^{18} + \beta d_2^{18}\right), \tag{31}$$
$$n_{31} = \alpha\left(d_1^{19} + \beta d_2^{19}\right), \tag{32}$$
$$n_{32} = \alpha\left(d_1^{20} + \beta d_2^{20}\right), \tag{33}$$
$$n_{33} = \alpha\left(d_1^{21} + \beta d_2^{21}\right), \tag{34}$$
$$m_{31} = \alpha\left(d_1^{22} + \beta d_2^{22}\right), \tag{35}$$
$$m_{32} = \alpha\left(d_1^{23} + \beta d_2^{23}\right), \tag{36}$$
$$m_{33} = \alpha\left(d_1^{24} + \beta d_2^{24}\right). \tag{37}$$

Combining (14), (16), (20), and (22), we obtain

$$n_{31}A_{31}m_{33} - n_{31}A_{33}m_{31} - n_{33}A_{11}m_{33} + n_{33}A_{13}m_{31} = 0. \tag{38}$$

Rewriting (38) in terms of elements of $\mathbf{d}_1$ and $\mathbf{d}_2$ gives

$$\begin{aligned}
&\alpha^3\left(d_1^{19} + \beta d_2^{19}\right)\left(d_1^7 + \beta d_2^7\right)\left(d_1^{24} + \beta d_2^{24}\right)\\
&- \alpha^3\left(d_1^{19} + \beta d_2^{19}\right)\left(d_1^9 + \beta d_2^9\right)\left(d_1^{22} + \beta d_2^{22}\right)\\
&- \alpha^3\left(d_1^{21} + \beta d_2^{21}\right)\left(d_1^1 + \beta d_2^1\right)\left(d_1^{24} + \beta d_2^{24}\right)\\
&+ \alpha^3\left(d_1^{21} + \beta d_2^{21}\right)\left(d_1^3 + \beta d_2^3\right)\left(d_1^{22} + \beta d_2^{22}\right) = 0.
\end{aligned} \tag{39}$$

Note that α is canceled out on both sides of (39) and β is the only variable involved in the final polynomial equation, which can be simply solved using Matlab.

Substituting the obtained β in (8), $\mathbf{W}$ is reformulated as $\mathbf{W} = \alpha\mathbf{d}$, where $\mathbf{d} = \mathbf{d}_1 + \beta\mathbf{d}_2 = (d_1,\ d_2,\ \cdots,\ d_{24})^{\mathrm{T}}$. The solution of $\mathbf{W}$ is in the solution space spanned by only one basis.

If $m_{31} \neq 0$, (14), (17), (20), (20), (23), (26), and (29) are reformatted as

$$n_{11} = \frac{-A_{11} + n_{31}m_{11}}{m_{31}} = \frac{-d_1 + d_{19}m_{11}}{d_{22}}, \tag{40}$$

$$n_{12} = \frac{-A_{21} + n_{32}m_{11}}{m_{31}} = \frac{-d_4 + d_{23}m_{11}}{d_{22}}, \tag{41}$$

$$n_{13} = \frac{-A_{31} + n_{33}m_{11}}{m_{31}} = \frac{-d_7 + d_{21}m_{11}}{d_{22}}, \tag{42}$$

$$n_{21} = \frac{-B_{11} + n_{31}m_{21}}{m_{31}} = \frac{-d_{10} + d_{19}m_{21}}{d_{22}}, \tag{43}$$

$$n_{22} = \frac{-B_{21} + n_{32}m_{21}}{m_{31}} = \frac{-d_{13} + d_{20}m_{21}}{d_{22}}, \tag{44}$$

$$n_{23} = \frac{-B_{31} + n_{33}m_{21}}{m_{31}} = \frac{-d_{16} + d_{21}m_{21}}{d_{22}}. \tag{45}$$

The first two columns of rotation matrices $\mathcal{M}, \mathcal{N}$ satisfy

$$\begin{cases} n_{11}^2 + n_{21}^2 + n_{31}^2 = 1 \\ m_{11}^2 + m_{21}^2 + m_{31}^2 = 1 \end{cases}. \tag{46}$$

Let $a = \frac{n_{31}}{m_{31}} = \frac{d_{19}}{d_{22}}$. Substituting (40), (43) and a into (46) gives

$$\left(\frac{-d_1 + d_{19}m_{11}}{d_{22}}\right)^2 + \left(\frac{-d_{10} + d_{19}m_{21}}{d_{22}}\right)^2 - a^2\left(m_{11}^2 + m_{21}^2\right) - 1 + a^2 = 0. \tag{47}$$

From (47), we can obtain

$$m_{21} = \frac{\left(d_1^2 - 2d_1d_{19}m_{11} + d_{10}^2 - d_{22}^2 + d_{19}^2\right)}{2d_{10}d_{19}}. \tag{48}$$

Substitution (48) in (43) and (44), n_{21} and n_{22} can now be expressed in terms of m_{11}. According to the orthogonality of the first two columns of $\mathcal{N}$, we have the following constraint

$$n_{11}n_{12} + n_{21}n_{22} + n_{31}n_{32} = 0. \tag{49}$$

Since (49) has only two unknowns α and m_{11}, we can then represent m_{11} in terms of α. Let m_{11}^{α} be m_{11} represented in terms of α, which we can obtain by the Matlab command: $m_{11}^{\alpha} = \text{solve}\left(n_{11}n_{12} + n_{21}n_{22} + \alpha^2 d_{19}d_{20}, m_{11}\right)$. We can similarly obtain n_{11}^{α}, n_{12}^{α}, n_{13}^{α}, m_{21}^{α}, n_{21}^{α}, n_{22}^{α}, and n_{23}^{α}. Since the second column of $\mathcal{N}$ satisfies $n_{12}^2 + n_{22}^2 + n_{32}^2 = 1$, α can be solved by $\alpha_{sol} = \text{solve}\left(\left(n_{12}^{\alpha}\right)^2 + \left(n_{22}^{\alpha}\right)^2 + \alpha^2 d_{20}^2 - 1, \alpha\right)$. Given α and β, $\mathbf{W}$ can then be obtained. The relative poses for the reference plane are then extracted.

If $m_{31} = 0$, the above derivation can be further simplified to obtain the results similarly.